Math 1510D Week 1

Set

A set is a collection of elements

eg B = the set of first 4 even numbers

a set

$= \{2, 4, 6, 8\}$ ← same

$= \{x : \underbrace{x \text{ is even}, 0 < x < 10}_{\text{condition}}\}$

such that

Notations

$x \in A$ means x is an element of A

$x \notin A$ means x is not an element of A

$A \subseteq B$ means A is a subset of B

(Every element of A is an element of B)

$A \nsubseteq B$ means A is not a subset of B

eg $A = \{2, 4, 6, 8\}$ $B = \{2, 8\}$ $C = \{2, 4\}$

$\therefore$ $8 \in A, B$

$B \subseteq A$, $C \subseteq A$, $B \nsubseteq C$

Set Operations

Let A, B be sets

Intersection

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

= the set of x such that x is in A and in B

Union

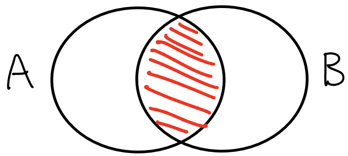
$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Relative Complement of B in A

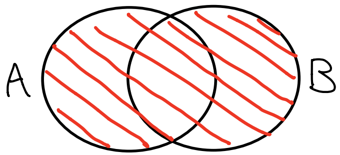
$$A \setminus B = \{x \in A : x \notin B\}$$

= the set of all x in A such that x is not in B

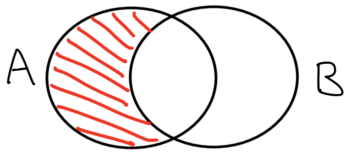
Picture (Venn Diagram)



$A \cap B$



$A \cup B$



$A \setminus B$

eg $A = \{2, 4, 6\}$ $B = \{3, 6, 9\}$

then $A \cap B = \{6\}$

$A \cup B = \{2, 3, 4, 6, 9\}$

$A \setminus B = \{2, 4\}$

Some important Sets

$\mathbb{N}$ = the set of all natural numbers

$= \{1, 2, 3, 4, 5, \cdots\}$

$\mathbb{Z}$ = the set of all integers

$= \{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$

$= \{0, \pm 1, \pm 2, \pm 3, \pm 4, \cdots\}$

$\mathbb{Q}$ = the set of all rational numbers

$\left\{ \frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}$

$\mathbb{R}$ = the set of all real numbers

eg $\frac{2}{3} \in \mathbb{Q}$ but $\frac{2}{3} \notin \mathbb{Z}$

$\sqrt{2} \in \mathbb{R}$, $\sqrt{2} \notin \mathbb{Q}$

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Intervals Let $a, b \in \mathbb{R}$ or $\pm\infty$

Open interval (Endpoints not included)

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

Closed interval (Endpoints included)

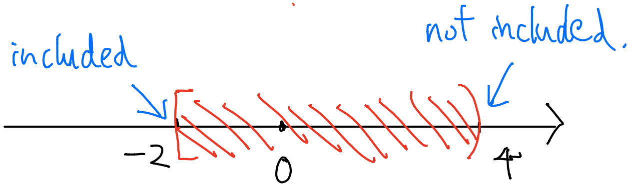
$$[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$$

Half-open interval

$$(a, b] = \{x \in \mathbb{R} : a < x \le b\}$$

$$[a, b) = \{x \in \mathbb{R} : a \le x < b\}$$

Picture $[-2, 4)$

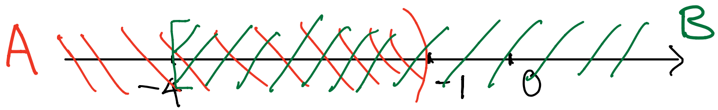


Ex Let $A = (-\infty, -1)$, $B = [-4, \infty)$

Express the followings as interval

i $A \cup B$ ii $A \cap B$ iii $A \setminus B$

Ans $A \cup B = (-\infty, \infty)$, $A \cap B = [-4, -1)$, $A \setminus B = (-\infty, -4)$



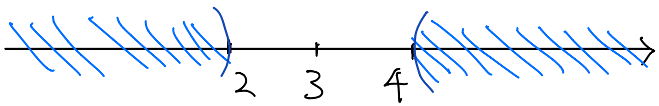
Ex Express $C = \{x \in \mathbb{R} : |x - 3| > 1\}$ as a union of intervals

Sol $|x-3| > 1$

$\Rightarrow \quad x - 3 > 1 \quad$ or $\quad x - 3 < -1$

$\Rightarrow \quad x > 4 \qquad\qquad x < 2$

$\Rightarrow \quad C = (-\infty, 2) \cup (4, \infty)$



Function

Let A, B be sets

A function $f: A \to B$ is a rule of assigning to each element of A an element of B

$A =$ Domain of f (Set of inputs)

$B =$ Codomain of f (A set containing all outputs)

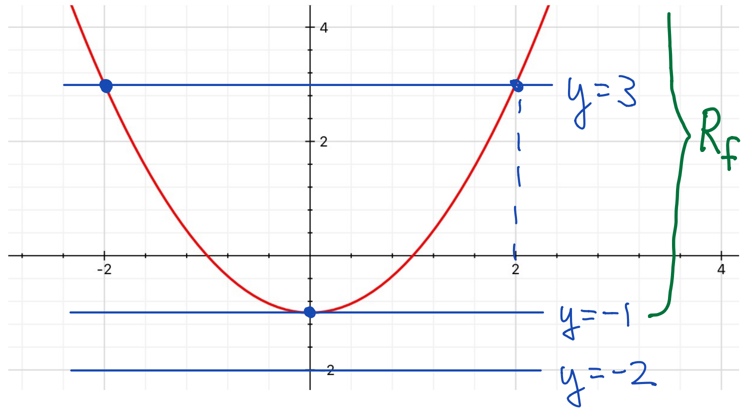
$R_f =$ Range of f (Set of outputs)

$= \{ f(x) \in B : x \in A \}$

Other notations

$D_f =$ domain of f

eg $\underbrace{f}_{} : \underset{\substack{\uparrow \\ \text{domain}}}{\mathbb{R}} \to \underset{\substack{\uparrow \\ \text{codomain}}}{\mathbb{R}}$ with $\underbrace{f(x) = x^2 - 1}_{\text{"rule of assignment"}}$



(no intersection)

$f(0) = -1 \Rightarrow -1 \in R_f$

$f(2) = 3 \Rightarrow 3 \in R_f$

$f(x) \neq -2$ for any $x \in D_f$

$\because x^2 \geq 0$

$\therefore f(x) = x^2 - 1 \geq -1$ $\Rightarrow -2 \notin R_f$

$R_f = [-1, \infty)$

Implied domain

If a function f(x) is given by an expression without specifying its domain, then the domain will be assumed to be the largest subset of $\mathbb{R}$ such that the expression makes sense.

That domain is called the Implied domain (or natural domain)

Useful rules

1. Denominator $\neq 0$
2. For $\log g(x)$, need $g(x) > 0$
3. Let m be an positive even number

   For $\sqrt[m]{h(x)} = [h(x)]^{\frac{1}{m}}$,

   need $h(x) \geq 0$

Rmk For 3,

eg $m = 3$ (odd)

$$64^{\frac{1}{3}} = 4$$

$$(-64)^{\frac{1}{3}} = -4$$

No problem

eg $m = 4$ (even)

$$(64)^{\frac{1}{4}} = 8^{\frac{1}{2}}$$

$(-64)^{\frac{1}{4}}$ is not real!

(fourth root of negative number)

eg Find implied domain of

a. $\log(x^2 - 3x - 10)$

b. $\dfrac{x-3}{\sqrt[4]{3 - |x|}}$

c. $(x+2)^{\frac{2}{3}}$

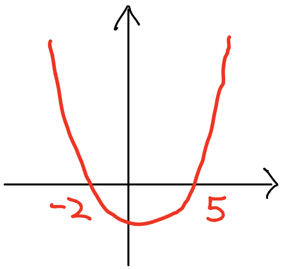
d. $f - g$, where $f(x) = \dfrac{1}{1+x}$ $g(x) = \dfrac{1}{1-x}$

Sol

a. Need $x^2 - 3x - 10 > 0$

$(x-5)(x+2) > 0$

$\therefore$ $x > 5$ or $x < -2$



$\Rightarrow$ Implied domain $= (-\infty, -2) \cup (5, \infty)$

b. Need $3 - |x| \geq 0$ under $\sqrt[4]{}$

Also, $\sqrt[4]{3-|x|} \neq 0$

$\Rightarrow$ $3 - |x| > 0$

$\Rightarrow$ $3 > |x|$

$\Rightarrow$ $-3 < x < 3$

Implied domain $= (-3, 3)$

c. $(x+2)^{\frac{2}{3}} = \sqrt[3]{(x+2)^2}$

3 is odd!

Addition, square, cubic root are defined for any real numbers

Implied domain $= \mathbb{R} = (-\infty, \infty)$

d. $(f-g)(x) = f(x) - g(x)$

$$= \underbrace{\frac{1}{1+x}}_{x \neq -1} - \underbrace{\frac{1}{1-x}}_{x \neq 1}$$

$\Rightarrow$ Implied domain $= \mathbb{R} \setminus \{\pm 1\}$

$= (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Operations on functions

Let $f(x), g(x)$ be functions. Define

$$(f \pm g)(x) = f(x) \pm g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{if } g(x) \neq 0$$

$$(g \circ f)(x) = g(f(x)) \quad \text{(Composition)}$$

$$x \xmapsto{f} f(x) \xmapsto{g} (g \circ f)(x)$$

$$\boxed{\begin{aligned} & D_{f+g} = D_{f-g} = D_{fg} = D_f \cap D_g \\ & D_{f/g} = (D_f \cap D_g) \setminus \{x \in D_g : g(x) = 0\} \\ & D_{g \circ f} = \{x \in D_f : f(x) \in D_g\} \end{aligned}}$$

eg Let $f(x) = x^2 - x$, $g : (2, \infty) \to \mathbb{R}$

a. Find $(f \circ f)(3)$.

b. Find the implied domain of $g \circ f$.

Sol

a. $(f \circ f)(3) = f(f(3)) = f(6) = 30$

b. $g \circ f(x) = g(f(x)) = g(x^2 - x)$

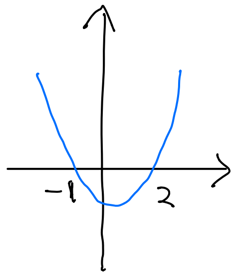
$$D_g = (2, \infty) \Rightarrow x^2 - x \in (2, \infty)$$

$$\Rightarrow x^2 - x > 2$$

$$\Rightarrow x^2 - x - 2 > 0$$

$$\Rightarrow (x-2)(x+1) > 0$$

$$\Rightarrow x > 2 \text{ or } x < -1$$



$$D_{g \circ f} = (-\infty, -1) \cup (2, \infty)$$

Inverse of a function

Suppose $f: A \to B$, $g: B \to A$

Then f, g are said to be inverse of each other if

for any

$(g \circ f)(x) = g(f(x)) = x \quad \forall x \in A$

$(f \circ g)(y) = f(g(y)) = y \quad \forall y \in B$

We write $f^{-1} = g$ and $g = f^{-1}$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

equal

$$\begin{tikzcd} x \arrow[r, mapsto] & f(x) \arrow[dl, mapsto] \\ g(f(x)) & \end{tikzcd}$$

$$\begin{tikzcd} & y \arrow[dl, mapsto] \\ g(y) \arrow[r, mapsto] & f(g(y)) \end{tikzcd}$$

equal

eg $f: [0, \infty) \to 0, \infty) \qquad f(x) = x^2$

$g: [0, \infty) \to 0, \infty) \qquad g(x) = \sqrt{x}$

$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = x$ for $x \geq 0$

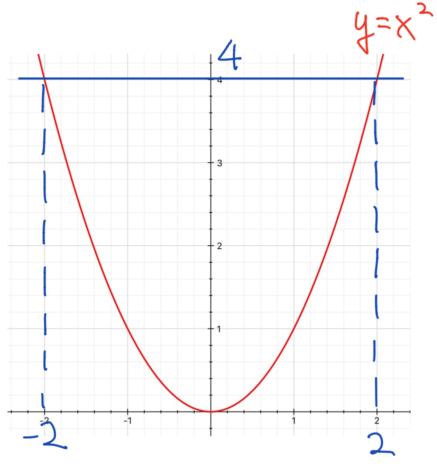
$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$ for $x \geq 0$

$\therefore f^{-1} = g \qquad g^{-1} = f$

Rmk

If $x < 0$, then

$$\begin{aligned} \sqrt{x^2} &= |x| \\ &= -x \\ &\neq x \end{aligned}$$

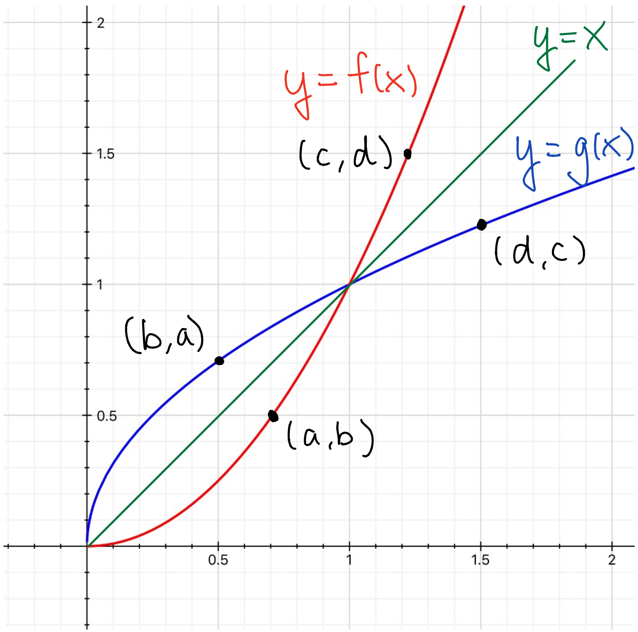


We need to be careful about domain when we talk about inverses

Graph of Inverse Functions

In last example $f(x) = x^2$

$g(x) = \sqrt{x}$



Graphs symmetric about $y = x$?

Note

(a, b) is on $y = f(x)$

$\iff b = f(a)$

$\iff a = g(b)$

$\iff (b, a)$ is on $y = g(x)$

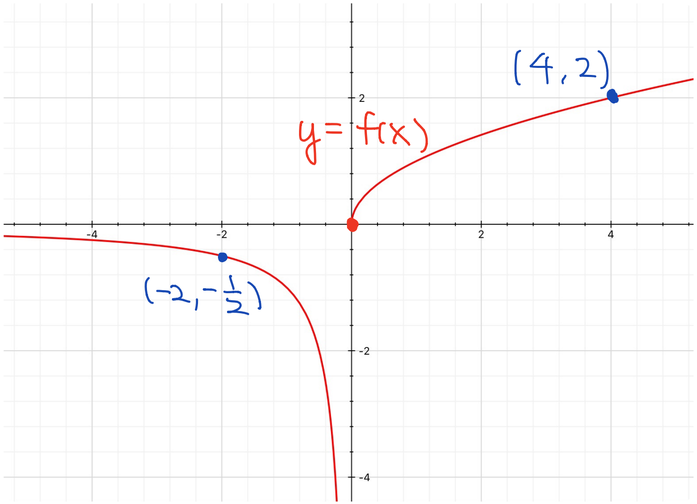
$$\boxed{\text{Graphs of inverse functions are mirror image of each other about the line } y = x}$$

Piecewise Function

eg

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ \frac{1}{x} & \text{if } x < 0 \end{cases}$$

- $f(4) = \sqrt{4} = 2$ $\quad (\because 4 \geq 0)$
- $f(-2) = \frac{1}{-2} = -\frac{1}{2}$ $\quad (\because -2 < 0)$



Q If $|h| < 1$, $f(1+h) = ?$

Note $-1 < h < 1 \Rightarrow 0 < 1+h < 2$

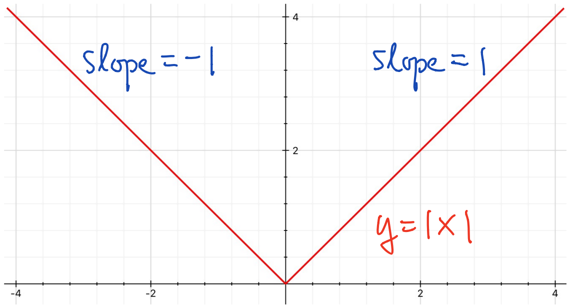
$\therefore f(1+h) = \sqrt{1+h}$

Absolute Value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

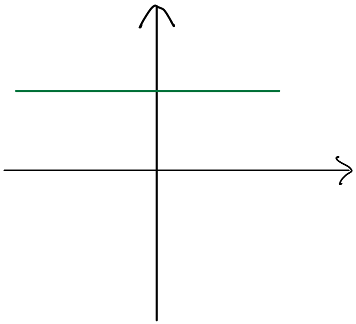
eg. $3 \geq 0 \Rightarrow |3| = 3$

$-2 < 0 \Rightarrow |-2| = -(-2) = 2$

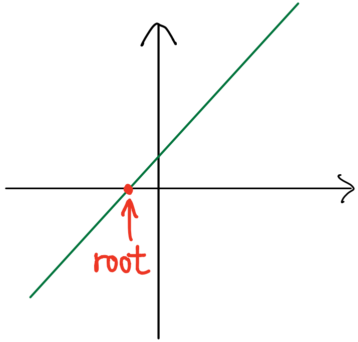


Graphs of some Elementary Functions

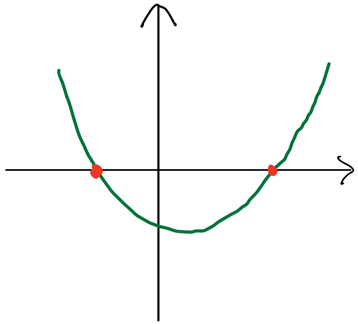
Polynomials $\quad f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = \sum_{i=0}^{n} a_i x^i$ (If $a_n \neq 0$, then $\deg f = n$)



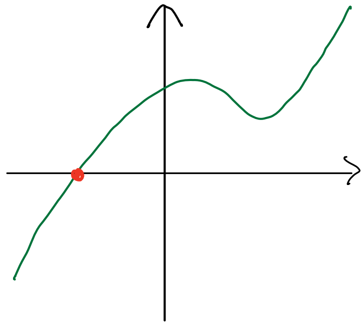
$\deg f = 0$
(constant)



$\deg f = 1$
(linear)



$\deg f = 2$
(quadratic)



$\deg f = 3$
(cubic)

> **Fact** A non-zero polynomial of degree n has at most n real roots

Exponential Functions

$f(x) = a^x$, where $a > 0$

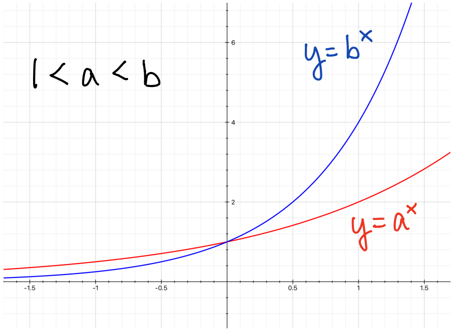
x ← variable

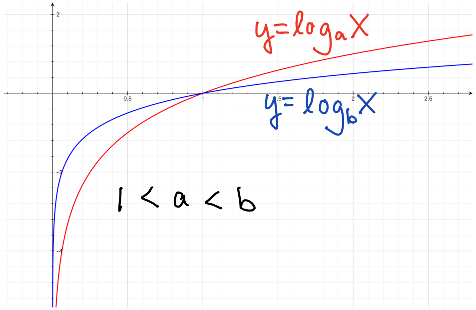
a ← constant

Logarithmic Functions

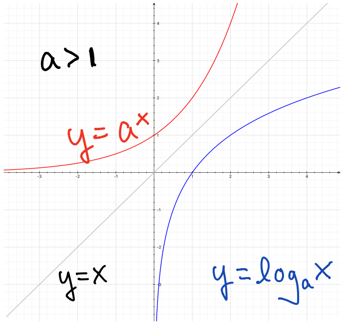
$f(x) = \log_a x$, where $a > 0$

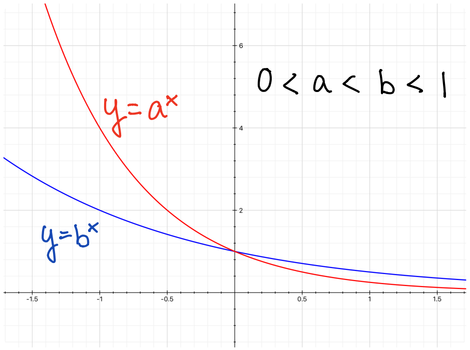
Inverse of each other

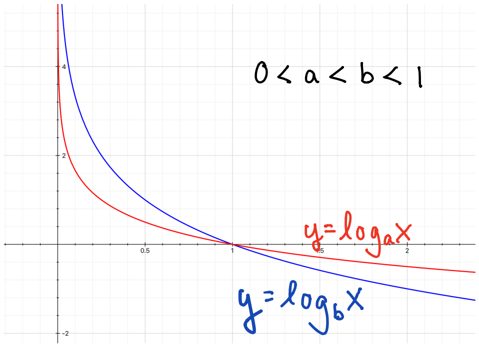




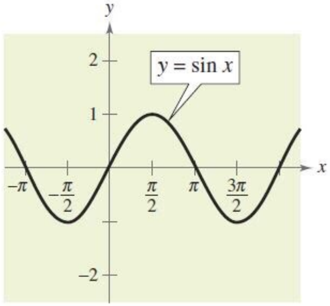
Symmetric about y=x



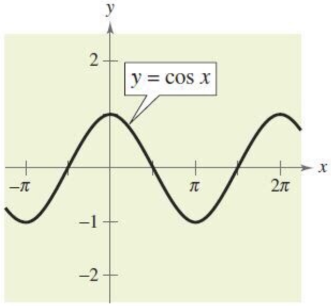




Trigonometric Functions



DOMAIN: $(-\infty, \infty)$
RANGE: $[-1, 1]$
PERIOD: 2π



DOMAIN: $(-\infty, \infty)$
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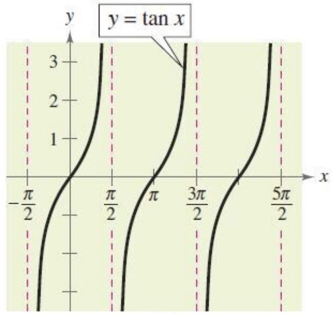
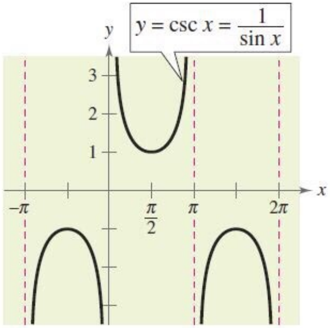
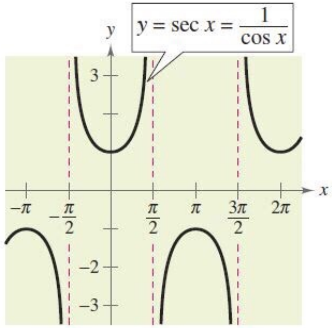


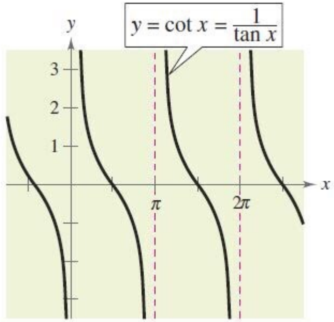
DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$
RANGE: $(-\infty, \infty)$
PERIOD: π



DOMAIN: ALL $x \neq n\pi$
RANGE: $(-\infty, -1] \cup [1, \infty)$
PERIOD: 2π



DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$
RANGE: $(-\infty, -1] \cup [1, \infty)$
PERIOD: 2π



DOMAIN: ALL $x \neq n\pi$
RANGE: $(-\infty, \infty)$
PERIOD: π

Rmk All "angles" here are measured in radian ($180^\circ = \pi$ rad)

Graphing a vector-valued function

We can graph

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

on the xy-plane :

$x(t)$, $y(t)$ are called components of $\vec{r}(t)$.

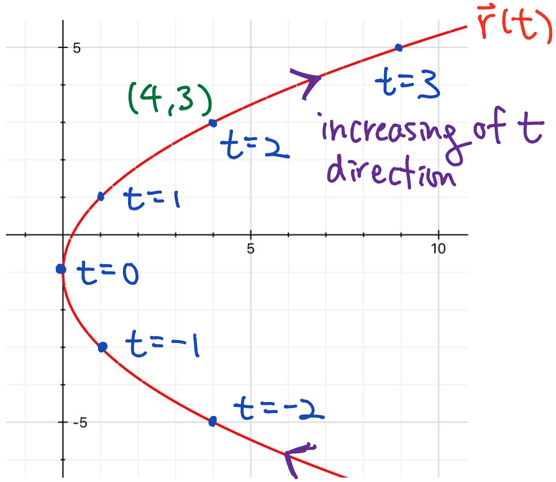
t is called parameter

eg $\vec{r}(t) = t^2\hat{i} + (2t-1)\hat{j}$, then

$$\begin{cases} x(t) = t^2 \\ y(t) = 2t-1 \end{cases}$$

called parametric equations

t	-2	-1	0	1	2
$x(t)$	4	1	0	1	4
$y(t)$	-5	-3	-1	1	3



Q How to plot $\vec{r}(t)$ above accurately ?

A Note $x = x(t) = t^2$

$$y = y(t) = 2t-1 \Rightarrow t = \frac{y+1}{2}$$

$$\therefore x = \left(\frac{y+1}{2}\right)^2 = \frac{1}{4}(y+1)^2 \quad \text{(Parabola)}$$

Q Graph $\vec{r}(t) = (2\cos t^\circ)\hat{i} + (2\sin t^\circ)\hat{j}$ for $0 \le t \le 180$.

Sol $x = 2\cos t^\circ$ $y = 2\sin t^\circ$

$\therefore x^2 + y^2 = (2\cos t^\circ)^2 + (2\sin t^\circ)^2$

$= 4(\cos^2 t^\circ + \sin^2 t^\circ)$

$= 4$

$\therefore \vec{r}(t)$ lies on the circle $x^2 + y^2 = 4$

Also, as t increases from 0 to 180,

$x(t)$ decreases from 2 to -2

$y(t)$ increases from 0 to 2 and then decreases from 2 to 0

$\therefore$ Graph of

$\vec{r}(t) = (2\cos t^\circ)\hat{i} + (2\sin t^\circ)\hat{j}$